

# An Asymptotic Solution for the Main Problem

Javier Herrera-Montojo\*, Hodei Urrutxua† and Jesús Peláez‡

*Technical University of Madrid (UPM), ETSI Aeronáuticos*

*Space Dynamics Group-UPM, Pz Cardenal Cisneros 3, 28040 Madrid, Spain*

**In this paper, an analytical solution of the main problem, a satellite only perturbed by the  $J_2$  harmonic, is derived with the aid of perturbation theory and by using DROMO variables. The solution, which is valid for circular and elliptic orbits with generic eccentricity and inclination, describes the instantaneous time variation of all orbital elements, that is, the actual values of the osculating elements.**

## I. Introduction

In 1975 A. Deprit in reference [1](#) proposed a new set of elements to describe Keplerian motions subject to perturbing forces. That work was followed by other papers (see references [2–5](#)) that involved the use of Euler parameters (quaternions) in the propagation of perturbed Keplerian orbits. Quaternions had been widely used in the analysis of the attitude dynamics of a rigid body (see for example [6](#)), but they had not been commonly used in orbital mechanics. This approach was proposed for the first time, to the author's knowledge, in [1](#).

In this paper, an analytical solution of the main problem, a satellite only perturbed by the  $J_2$  harmonic, is derived with the aid of perturbation theory. The solution, which is valid for circular and elliptic orbits with generic eccentricity and inclination, describes the instantaneous time variation of all orbital elements.

This asymptotic solution includes the secular variation of the elements and also the short periodic time dependence, which exhibit an interesting structure showing that the main frequencies involved in the elliptic case are the angular frequency of the orbit and its first five multiples.

In year 2000 an in-house orbital propagator was developed by the SDG-UPM (former Grupo de Dinámica de Tethers). The aim of the project was to develop a regular, robust and efficient propagator. Regular means that the propagator is free of singularities. Robust means that it should be numerically stable. Efficient means that the propagator should render accurate results with low time-consumption and share a common formulation for elliptical, parabolic and hyperbolic problems. How accurate the results are and how quickly they are provided by the propagator constitute a metric of its performance.

To fulfill these requirements, a special perturbation method of variation of parameters (VOP) based in a set of redundant variables including quaternions was considered. That propagator was called DROMO and initially it was mainly used in numerical simulations of electrodynamic tethers.

The basic theory of DROMO can be found in references [7,8](#). DROMO is closely related to the *ideal frame* concept introduced by Hansen in the XIX century. The Hansen's ideal frame is attached to the orbital plane and allows us to separate the periodic perturbations in the orbital plane from those of the orbital plane itself. The other important and related feature of the ideal frame is that the equations of relative and absolute motions have the same form, because the fictitious forces of Coriolis and centrifugal are equal and opposed. The components of a quaternion describing the attitude of the ideal frame are used in DROMO as parameters.

In spite of DROMO has been directly formulated as a special perturbation method that gives its best in numerical simulations it has also been used to obtain asymptotic solutions in the two-body problem with constant tangential thrust acceleration (see reference [9](#)).

There are several interesting applications of an analytical solution like the one presented in this paper. Whenever a fast calculation of an orbit would be needed the analytical solution provides a very good first approximation.

## II. Review of DROMO formulation

### A. Scenario

A particle  $M$  is moving in a fixed inertial frame  $Ox_1y_1z_1$  (solid 1). The origin  $O$  is the center of mass of a celestial body.  $M$  is acted upon by:

- 1) the gravitation of the celestial body
- 2) the remaining forces included in the perturbing acceleration  $\vec{a}_p$ .

The governing equation is:

$$\frac{d^2 \vec{x}}{dt^2} = -\frac{\mu}{|\vec{x}|^3} \vec{x} + \vec{f}_p \quad (1)$$

which should be integrated from appropriated initial conditions:

$$\text{at } t = 0 : \quad \vec{x} = \vec{x}_0, \quad \vec{v} = \vec{v}_0 \quad (2)$$

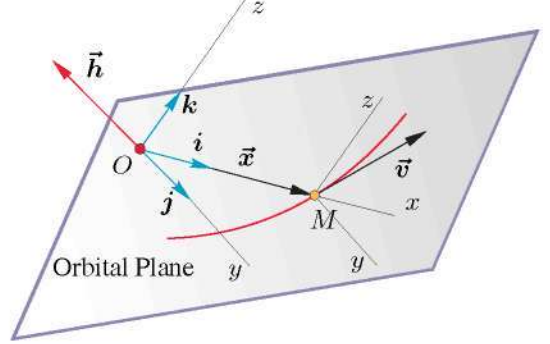


Figure 1. Orbital frame

In DROMO the orbital frame  $Mxyz$  plays a central role. However, it is more convenient to use the frame  $Oxyz$  (see figure 1) with origin in  $O$ . The planes  $Oxz$  and  $Mxz$  coincide from a geometrical point of view. However, they are different planes from a kinematical point of view. Which one is the real orbital plane? Later on we will give an answer.

In the pure Keplerian motion the orbital plane is situated by three Euler angles that take constant values:

- the longitude of the ascending node  $\Omega$ ,
- the inclination  $i$  and
- the argument of pericenter  $\omega$ .

These angles are associated with the perifocal frame  $Ox_py_pz_p$  at the initial instant: we call this frame the **departure perifocal frame**. Figure 2 shows a sketch of the departure perifocal frame  $Ox_py_pz_p$ . The point  $P$  of the figure coincides with the pericenter of the orbit at the initial time.

We assume that the particle  $M$  follows an elliptical orbit around an oblate planet. The unique perturbation which is acting on the particle is the one associated to the zonal harmonic  $J_2$ , that is, we face the *main problem*. We look for an analytical solution by using perturbation techniques and the formulation of DROMO. In this paper the orbital plane is attached to the departure perifocal frame  $Ox_py_pz_p$  and the motion of the satellite will be decomposed in the motion relative to the orbital plane plus the motion of the orbital plane.

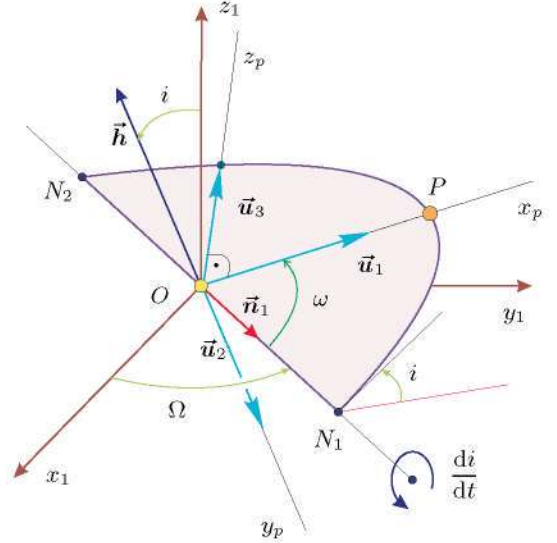


Figure 2. Departure perifocal frame

At the initial time the basis  $[i, j, k]$  of the orbital frame and  $[i_1, j_1, k_1]$  of the inertial frame are related by

$$[i, j, k] = [i_1, j_1, k_1] Q(0),$$

where the matrix  $Q(0)$  is:

$$Q(0) = \begin{pmatrix} +\cos \Omega \cos(\theta_0 + \omega) - \cos i \sin \Omega \sin(\theta_0 + \omega) & -\sin i \sin \Omega & -\cos \Omega \sin(\theta_0 + \omega) - \cos i \sin \Omega \cos(\theta_0 + \omega) \\ +\sin \Omega \cos(\theta_0 + \omega) + \cos i \cos \Omega \sin(\theta_0 + \omega) & +\sin i \cos \Omega & -\sin \Omega \sin(\theta_0 + \omega) + \cos i \cos \Omega \cos(\theta_0 + \omega) \\ +\sin i \sin(\theta_0 + \omega) & -\cos i & +\sin i \cos(\theta_0 + \omega) \end{pmatrix}$$

Here  $\Omega, i, \omega$  are classical elements of the orbit and  $\theta_0$  the initial true anomaly. Note that the dynamic state of the satellite, at  $t = 0$ , is given by  $\vec{x}_0$  and  $\vec{v}_0$ .

The frame  $Ox_py_pz_p$  is also inertial in a pure Keplerian motion. However, in a perturbed motion that frame will change with time. In a given time  $t$  is related with the inertial frame through the relations:

$$[i_p, j_p, k_p] = [i_1, j_1, k_1] P$$

At the initial time the matrix  $P(0)$  is given by:

$$P(0) = \begin{pmatrix} +\cos \Omega \cos \omega - \cos i \sin \Omega \sin \omega & -\sin i \sin \Omega & -\cos \Omega \sin \omega - \cos i \sin \Omega \cos \omega \\ +\sin \Omega \cos \omega + \cos i \cos \Omega \sin \omega & +\sin i \cos \Omega & -\sin \Omega \sin \omega + \cos i \cos \Omega \cos \omega \\ +\sin i \sin \omega & -\cos i & +\sin i \cos \omega \end{pmatrix}$$

and it can be obtained from the matrix  $Q(0)$  vanishing the angle  $\theta_0 = 0$ .

Matrix  $P$  evolves with time due to the perturbations. Associated to this matrix  $P$  we define an unitary quaternion  $\mathbf{p}$ . Its components are:

$$\mathbf{p} = \varepsilon_4^0 + (\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0)$$

and also change with time. They are related with the classical elements through the equations

$$\sin \frac{i}{2} \cos \frac{\Omega - \tilde{\omega}}{2} = \frac{\sqrt{2}}{2} (\varepsilon_1^0 + \varepsilon_4^0), \quad \cos \frac{i}{2} \sin \frac{\Omega + \tilde{\omega}}{2} = \frac{\sqrt{2}}{2} (\varepsilon_3^0 - \varepsilon_2^0) \quad (3)$$

$$\sin \frac{i}{2} \sin \frac{\Omega - \tilde{\omega}}{2} = \frac{\sqrt{2}}{2} (\varepsilon_2^0 + \varepsilon_3^0), \quad \cos \frac{i}{2} \cos \frac{\Omega + \tilde{\omega}}{2} = \frac{\sqrt{2}}{2} (\varepsilon_4^0 - \varepsilon_1^0) \quad (4)$$

where  $\tilde{\omega}$  coincides with  $\omega$  at the initial time but is different for later instants.

In a generic time  $t$  the unit vectors of the orbital frame  $Oxyz$  are given by

$$[i, j, k] = [i_1, j_1, k_1] Q(t),$$

where matrix  $Q(t)$  have associated a unitary quaternion  $\mathbf{q}$  whose components are:

$$\mathbf{q} = \varepsilon_4 + (\varepsilon_1, \varepsilon_2, \varepsilon_3)$$

In order to pass from the vectors  $[i_p, j_p, k_p]$  to the vectors  $[i, j, k]$  a rotation of intensity  $-\sigma$  around the axis  $Oy_p$  must be performed.

$$[i, j, k] = [\vec{i}_p, \vec{j}_p, \vec{k}_p] R_2(-\sigma) \quad \text{con} \quad R_2(-\sigma) = \begin{pmatrix} +\cos \sigma & 0 & -\sin \sigma \\ 0 & 1 & 0 \\ +\sin \sigma & 0 & +\cos \sigma \end{pmatrix}$$

As a consequence, matrices  $P$  and  $Q$  are related by equation:

$$Q = P R_2(-\sigma)$$

that can be translated to a relation between the corresponding quaternions:

$$\mathbf{q} = \mathbf{p} \cdot \mathbf{r}_2$$

where  $\mathbf{r}_2$  is the quaternion associated with matrix  $R_2(-\sigma)$  and whose components are:

$$\mathbf{r}_2 = \cos\left(\frac{\sigma}{2}\right) + (0, -\sin\left(\frac{\sigma}{2}\right), 0)$$

The product of quaternions provides the following matrix equation:

$$\begin{Bmatrix} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_2 \\ \varepsilon_4 \end{Bmatrix} = \begin{bmatrix} \cos \frac{\sigma}{2} & \sin \frac{\sigma}{2} & 0 & 0 \\ -\sin \frac{\sigma}{2} & \cos \frac{\sigma}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\sigma}{2} & -\sin \frac{\sigma}{2} \\ 0 & 0 & \sin \frac{\sigma}{2} & \cos \frac{\sigma}{2} \end{bmatrix} \begin{Bmatrix} \varepsilon_1^0 \\ \varepsilon_3^0 \\ \varepsilon_2^0 \\ \varepsilon_4^0 \end{Bmatrix} \quad (5)$$

giving the position of the orbital frame in terms of the  $\sigma$ -angle (ideal anomaly) and the position of the departure perifocal frame.

The DROMO theory developed in<sup>7,8</sup> can be summarized in the following set of equations:

$$\begin{aligned}\frac{d\tau}{d\sigma} &= \frac{1}{q_3 s^2} \\ \frac{dq_3}{d\sigma} &= -\frac{1}{s^3} \vec{a}_p \cdot \vec{k} \\ \frac{dq_1}{d\sigma} &= +\frac{\sin \sigma}{q_3 s^2} \vec{a}_p \cdot \vec{i} + \cos \sigma \frac{s + q_3}{q_3 s^3} \vec{a}_p \cdot \vec{k} \\ \frac{dq_2}{d\sigma} &= -\frac{\cos \sigma}{q_3 s^2} \vec{a}_p \cdot \vec{i} + \sin \sigma \frac{s + q_3}{q_3 s^3} \vec{a}_p \cdot \vec{k} \\ \frac{d\varepsilon_1^0}{d\sigma} &= -\frac{\lambda(\sigma)}{2} \{\sin(\sigma) \varepsilon_2^0 + \cos(\sigma) \eta^0\} \\ \frac{d\varepsilon_2^0}{d\sigma} &= +\frac{\lambda(\sigma)}{2} \{\sin(\sigma) \varepsilon_1^0 - \cos(\sigma) \varepsilon_3^0\} \\ \frac{d\varepsilon_3^0}{d\sigma} &= +\frac{\lambda(\sigma)}{2} \{\cos(\sigma) \varepsilon_2^0 - \sin(\sigma) \eta^0\} \\ \frac{d\eta^0}{d\sigma} &= +\frac{\lambda(\sigma)}{2} \{\cos(\sigma) \varepsilon_1^0 + \sin(\sigma) \varepsilon_3^0\}\end{aligned}$$

which must be integrated taking into account the following relations:

$$\begin{aligned}\lambda(\sigma) &= \frac{1}{q_3 s^3} \cdot \vec{a}_p \cdot \vec{j} \\ z &= \frac{1}{r} = q_3 \cdot s = q_3 \cdot \{q_3 + q_1 \cos \sigma + q_2 \sin \sigma\} \\ \frac{dr}{d\tau} &= -\psi \frac{dz}{d\sigma} = q_1 \sin \sigma - q_2 \cos \sigma \\ \left\{ \begin{array}{c} \varepsilon_1 \\ \varepsilon_3 \\ \varepsilon_2 \\ \eta \end{array} \right\} &= \left[ \begin{array}{cccc} \cos \frac{\sigma}{2} & \sin \frac{\sigma}{2} & 0 & 0 \\ -\sin \frac{\sigma}{2} & \cos \frac{\sigma}{2} & 0 & 0 \\ 0 & 0 & \cos \frac{\sigma}{2} & -\sin \frac{\sigma}{2} \\ 0 & 0 & \sin \frac{\sigma}{2} & \cos \frac{\sigma}{2} \end{array} \right] \left\{ \begin{array}{c} \varepsilon_1^0(\sigma) \\ \varepsilon_3^0(\sigma) \\ \varepsilon_2^0(\sigma) \\ \eta^0(\sigma) \end{array} \right\}\end{aligned}$$

In these non-dimensional equations  $\tau$  is the time,  $r$  the non-dimensional radial distance,  $\sigma$  the ideal anomaly, that is, the angle between the axes  $Ox$  —orbital frame— and  $Ox_p$  —departure perifocal frame— and the variable  $q_3$  is the inverse of the non-dimensional angular momentum. The eccentricity vector  $\vec{e}$  has the components:

$$\vec{e} = \frac{q_1}{q_3} \vec{i}_p + \frac{q_2}{q_3} \vec{k}_p$$

in the departure perifocal frame. Finally,  $\vec{a}_p$  is the non-dimensional perturbing acceleration acting on the particle  $M$ .

Obviously these equations must be integrated starting from the appropriate initial conditions:

$$\text{at } \sigma = \sigma_0 : \quad \tau = 0, \quad q_3(0) = 1/h_0, \quad q_2(0) = 0, \quad q_1(0) = e q_e(0), \quad \varepsilon_i^0 = \varepsilon_i^0(0)$$

that should be deduced from the initial vector state:  $\vec{x}_0$  and  $\vec{v}_0$  (see below the deduction of relation 10).

### III. Main problem in DROMO variables

In the main problem the perturbing acceleration derive from the following potential function:

$$V_p(\vec{x}) = -\mu J_2 \frac{R_E^2}{|\vec{x}|^3} \frac{1}{2} \{1 - 3 \sin^2 \varphi\}$$



where  $\varphi$  is the latitude of the satellite  $M$ ,  $R_E$  is the equatorial radius of the oblate planet and  $J_2$  the second zonal harmonic. Therefore the perturbing acceleration is given by

$$\vec{f}_p = -\nabla V_p = -\frac{\partial V_p}{\partial r} \vec{u}_r - \frac{1}{r} \frac{\partial V_p}{\partial \varphi} \vec{u}_\varphi = -\frac{3\mu J_2 R_E^2}{2r^4} \{(1 - 3\sin^2 \varphi) \vec{u}_r + \sin 2\varphi \vec{u}_\varphi\}$$

From this expression it is not difficult to obtain the components of the perturbing acceleration in the orbital frame:

$$\begin{aligned}\vec{f}_p \cdot \vec{i} &= -\frac{3\mu}{R^2} J_2 \left(\frac{R_E}{R}\right)^2 \left[\frac{1}{2} - \frac{3}{2}(\vec{i} \cdot \vec{k}_1)^2\right] \\ \vec{f}_p \cdot \vec{j} &= -\frac{3\mu}{R^2} J_2 \left(\frac{R_E}{R}\right)^2 (\vec{i} \cdot \vec{k}_1)(\vec{j} \cdot \vec{k}_1) \\ \vec{f}_p \cdot \vec{k} &= -\frac{3\mu}{R^2} J_2 \left(\frac{R_E}{R}\right)^2 (\vec{i} \cdot \vec{k}_1)(\vec{k} \cdot \vec{k}_1)\end{aligned}$$

where  $R = |\vec{x}|$ .

### Non-dimensional variables

The perturbing acceleration can be reduced to non-dimensional form as follows:

$$\vec{f}_p = L_c \omega_0^2 \vec{a}_p, \quad \text{with} \quad \omega_0^2 = \frac{\mu}{L_c^3}$$

and where  $L_c$  is a characteristic length. Thus, the non-dimensional components of the perturbing acceleration are:

$$\begin{aligned}\vec{a}_p \cdot \vec{i} &= -3\epsilon z^4 \left[\frac{1}{2} - \frac{3}{2}(\vec{i} \cdot \vec{k}_1)^2\right] \\ \vec{a}_p \cdot \vec{j} &= -3\epsilon z^4 (\vec{i} \cdot \vec{k}_1)(\vec{j} \cdot \vec{k}_1) \\ \vec{a}_p \cdot \vec{k} &= -3\epsilon z^4 (\vec{i} \cdot \vec{k}_1)(\vec{k} \cdot \vec{k}_1)\end{aligned}$$

where

$$\epsilon = J_2 \left(\frac{R_E}{L_c}\right)^2, \quad \text{and} \quad z = \frac{1}{r} = q_3 \cdot s = q_3 \cdot (q_3 + q_1 \cos \sigma + q_2 \sin \sigma)$$

It should be noticed that the right hand side of these equations should be expressed in terms of the DROMO state variables:

$$\sigma, q_1, q_2, q_3, \varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0, \varepsilon_4^0$$

Let  $(\vec{i}_1, \vec{j}_1, \vec{k}_1)$  be the unit vectors of the inertial frame  $O_1x_1y_1z_1$ ; the unit vectors of the orbital frame  $(\vec{i}, \vec{j}, \vec{k})$  are given by the *vectrix* equation:

$$[\vec{i}, \vec{j}, \vec{k}] = [\vec{i}_1, \vec{j}_1, \vec{k}_1] Q(t)$$

where  $Q$  is an orthogonal matrix. To describe the time evolution of matrix  $Q$  we use the Euler parameters  $(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4)$  which can be grouped in a vector of the orbital frame:

$$\vec{\varepsilon} = \varepsilon_1 \vec{i} + \varepsilon_2 \vec{j} + \varepsilon_3 \vec{k}$$

and an independent scalar  $\varepsilon_4$ .

The matrix  $Q$  in terms of the Euler parameter is given by:

$$Q = \begin{bmatrix} 1 - 2(\varepsilon_2^2 + \varepsilon_3^2) & 2\varepsilon_1\varepsilon_2 - 2\varepsilon_4\varepsilon_3 & 2\varepsilon_1\varepsilon_3 + 2\varepsilon_4\varepsilon_2 \\ 2\varepsilon_1\varepsilon_2 + 2\varepsilon_4\varepsilon_3 & 1 - 2(\varepsilon_1^2 + \varepsilon_3^2) & 2\varepsilon_2\varepsilon_3 - 2\varepsilon_4\varepsilon_1 \\ 2\varepsilon_1\varepsilon_3 - 2\varepsilon_4\varepsilon_2 & 2\varepsilon_3\varepsilon_2 + 2\varepsilon_4\varepsilon_1 & 1 - 2(\varepsilon_1^2 + \varepsilon_2^2) \end{bmatrix}$$

Therefore, the scalar products involved in the components of  $\vec{a}_p$  are:

$$\begin{aligned}\vec{i} \cdot \vec{k}_1 &= 2(\varepsilon_1 \varepsilon_3 - \varepsilon_2 \varepsilon_4) \\ \vec{j} \cdot \vec{k}_1 &= 2(\varepsilon_2 \varepsilon_3 + \varepsilon_1 \varepsilon_4) \\ \vec{k} \cdot \vec{k}_1 &= 1 - 2(\varepsilon_1^2 + \varepsilon_2^2)\end{aligned}$$

However, the Euler parameters of the frame  $Oxyz$  must be expressed in terms of the DROMO state variables, that is, in terms of the Euler parameters of the departure perifocal frame  $\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0, \varepsilon_4^0$ . Both sets of Euler parameters are related by equations (5). Thus we arrive to the following result:

$$\begin{aligned}\mathbf{i} \cdot \mathbf{k}_1 &= -[1 - 2\{(\varepsilon_3^0)^2 + (\varepsilon_4^0)^2\}] \sin \sigma + 2[\varepsilon_1^0 \varepsilon_3^0 - \varepsilon_2^0 \varepsilon_4^0] \cos \sigma \\ \mathbf{j} \cdot \mathbf{k}_1 &= 2(\varepsilon_2^0 \varepsilon_3^0 + \varepsilon_1^0 \varepsilon_4^0) \\ \mathbf{k} \cdot \mathbf{k}_1 &= -[1 - 2\{(\varepsilon_3^0)^2 + (\varepsilon_4^0)^2\}] \cos \sigma - 2[\varepsilon_1^0 \varepsilon_3^0 - \varepsilon_2^0 \varepsilon_4^0] \sin \sigma\end{aligned}$$

which permit to express the right hand side of the DROMO governing equations in terms of the DROMO state variables.

### Governing equations

The above analysis leads to the following set of equations which should be integrated:

$$\begin{aligned}\frac{d\tau}{d\sigma} &= \frac{1}{q_3 s^2} \\ \frac{dq_1}{d\sigma} &= -\epsilon \cdot 3q_3^3 \left\{ +s \sin \sigma \left[ \frac{1}{2} - \frac{3}{2}(\mathbf{i} \cdot \mathbf{k}_1)^2 \right] + (s + q_3) \cos \sigma (\mathbf{i} \cdot \mathbf{k}_1)(\mathbf{k} \cdot \mathbf{k}_1) \right\} \\ \frac{dq_2}{d\sigma} &= -\epsilon \cdot 3q_3^3 \left\{ -s \cos \sigma \left[ \frac{1}{2} - \frac{3}{2}(\mathbf{i} \cdot \mathbf{k}_1)^2 \right] + (s + q_3) \sin \sigma (\mathbf{i} \cdot \mathbf{k}_1)(\mathbf{k} \cdot \mathbf{k}_1) \right\} \\ \frac{dq_3}{d\sigma} &= +\epsilon \cdot 3q_3^4 s (\mathbf{i} \cdot \mathbf{k}_1)(\mathbf{k} \cdot \mathbf{k}_1) \\ \frac{d\varepsilon_1^0}{d\sigma} &= +\epsilon \cdot \frac{3}{2} q_3^3 s \{ \sin \sigma \varepsilon_2^0 + \cos \sigma \varepsilon_4^0 \} \\ \frac{d\varepsilon_2^0}{d\sigma} &= -\epsilon \cdot \frac{3}{2} q_3^3 s \{ \sin \sigma \varepsilon_1^0 - \cos \sigma \varepsilon_3^0 \} \\ \frac{d\varepsilon_3^0}{d\sigma} &= -\epsilon \cdot \frac{3}{2} q_3^3 s \{ \cos \sigma \varepsilon_2^0 - \sin \sigma \varepsilon_4^0 \} \\ \frac{d\varepsilon_4^0}{d\sigma} &= -\epsilon \cdot \frac{3}{2} q_3^3 s \{ \cos \sigma \varepsilon_1^0 + \sin \sigma \varepsilon_3^0 \}\end{aligned}$$

where  $s = q_3 + q_1 \cos \sigma + q_2 \sin \sigma$  and the scalar products involved in the right hand sides are given by:

$$\begin{aligned}\mathbf{i} \cdot \mathbf{k}_1 &= -[1 - 2\{(\varepsilon_3^0)^2 + (\varepsilon_4^0)^2\}] \sin \sigma + 2[\varepsilon_1^0 \varepsilon_3^0 - \varepsilon_2^0 \varepsilon_4^0] \cos \sigma \\ \mathbf{j} \cdot \mathbf{k}_1 &= 2(\varepsilon_2^0 \varepsilon_3^0 + \varepsilon_1^0 \varepsilon_4^0) \\ \mathbf{k} \cdot \mathbf{k}_1 &= -[1 - 2\{(\varepsilon_3^0)^2 + (\varepsilon_4^0)^2\}] \cos \sigma - 2[\varepsilon_1^0 \varepsilon_3^0 - \varepsilon_2^0 \varepsilon_4^0] \sin \sigma\end{aligned}$$

In the integration the following relations must be taken into account:

$$\begin{aligned}\lambda(\sigma) &= \frac{P}{Q} = \frac{1}{q_3 s^3} \cdot (\vec{\mathbf{f}}_p \cdot \mathbf{j}) \\ z &= \frac{1}{r} = q_3 \cdot s = q_3 \cdot \{q_3 + q_1 \cos \sigma + q_2 \sin \sigma\} \\ \frac{dr}{d\tau} &= -\psi \frac{dz}{d\sigma} = q_1 \sin \sigma - q_2 \cos \sigma \\ 1 &= (\varepsilon_1^0)^2 + (\varepsilon_2^0)^2 + (\varepsilon_3^0)^2 + (\varepsilon_4^0)^2\end{aligned}$$

together with equations (5)

### Initial conditions

To propagate the perturbed orbit the initial dynamic state—which is defined by  $\vec{\mathbf{x}}_0$  and  $\vec{\mathbf{v}}_0$ —is a known data. The values of  $L_c$  and  $\omega_0$  used to obtain the non-dimensional equations also are assumed known. Finally at the initial instant we consider  $\tau = 0$ .

1. the value  $\sigma_0 = \theta_0$  coincides with the **initial true anomaly** and it can be calculated in a classical way from the initial data  $\vec{x}_0$  and  $\vec{v}_0$
2. the values  $r_0$  —initial value of de  $r$  and non-dimensional— and the initial angular momentum  $h_0$  turn out to be:

$$r_0 = \frac{|\vec{x}_0|}{L_c} \quad h_0 = \frac{|\vec{x}_0 \times \vec{v}_0|}{\omega_0 L_c^2} \quad \Rightarrow \quad q_3(0) = \frac{1}{h_0}, \quad s_0 = \frac{h_0}{r_0}$$

3. the initial velocity vector takes the form

$$\vec{v}_0 = \omega_0 L_c \left\{ \frac{dr}{d\tau} \Big|_0 \mathbf{i} + \frac{h}{r} \mathbf{k} \right\} = \omega_0 L_c \left\{ \frac{dr}{d\tau} \Big|_0 \mathbf{i} + s_0 \mathbf{k} \right\}$$

The radial velocity is given by:

$$\dot{r}_0 = \frac{dr}{d\tau} \Big|_0 = \frac{\vec{x}_0 \cdot \vec{v}_0}{r_0 \omega_0 L_c^2} = \frac{\vec{x}_0 \cdot \vec{v}_0}{|\vec{x}_0| \omega_0 L_c}$$

4. the initial eccentricity vector  $\vec{e}_0$  takes the value:

$$\vec{e}_0 = -\mathbf{i} + h_0 \mathbf{j} \times \left\{ \frac{dr}{d\tau} \Big|_0 \mathbf{i} + s_0 \mathbf{k} \right\} = (h_0 s_0 - 1) \mathbf{i} - h_0 \dot{r}_0 \mathbf{k}$$

As a consequence the eccentricity turns out to be:

$$e_0 = \sqrt{(h_0 s_0 - 1)^2 + h_0^2 \dot{r}_0^2}$$

To complete the calculations of the initial conditions, note that the relations

$$\begin{aligned} \mathbf{i} &= +\cos \sigma_0 \mathbf{i}_p + \sin \sigma_0 \mathbf{k}_p \\ \mathbf{k} &= -\sin \sigma_0 \mathbf{i}_p + \cos \sigma_0 \mathbf{k}_p \\ \dot{r}_0 &= q_{1_0} \sin \sigma_0 - q_{2_0} \cos \sigma_0 \\ s_0 &= q_{3_0} + q_{1_0} \cos \sigma_0 + q_{2_0} \sin \sigma_0 \end{aligned}$$

permit to express the initial eccentricity vector as follows:

$$\vec{e}_0 = \frac{q_1(0)}{q_3(0)} \mathbf{i}_p + \frac{q_2(0)}{q_3(0)} \mathbf{k}_p = e_0 \mathbf{i}_p$$

and therefore:

$$q_1(0) = e_0 q_3(0), \quad q_2(0) = 0$$

5. the **initial values** of the components of the quaternion  $(\varepsilon_1^0, \varepsilon_2^0, \varepsilon_3^0, \varepsilon_4^0)$  are:

$$\varepsilon_1^0(0) = \frac{\sqrt{2}}{2} \left( \sin \frac{i_0}{2} \cos \frac{\Omega_0 - \omega_0}{2} - \cos \frac{i_0}{2} \cos \frac{\Omega_0 + \omega_0}{2} \right) \quad (6)$$

$$\varepsilon_2^0(0) = \frac{\sqrt{2}}{2} \left( \sin \frac{i_0}{2} \sin \frac{\Omega_0 - \omega_0}{2} - \cos \frac{i_0}{2} \sin \frac{\Omega_0 + \omega_0}{2} \right) \quad (7)$$

$$\varepsilon_3^0(0) = \frac{\sqrt{2}}{2} \left( \sin \frac{i_0}{2} \sin \frac{\Omega_0 - \omega_0}{2} + \cos \frac{i_0}{2} \sin \frac{\Omega_0 + \omega_0}{2} \right) \quad (8)$$

$$\varepsilon_4^0(0) = \frac{\sqrt{2}}{2} \left( \sin \frac{i_0}{2} \cos \frac{\Omega_0 - \omega_0}{2} + \cos \frac{i_0}{2} \cos \frac{\Omega_0 + \omega_0}{2} \right) \quad (9)$$

where  $(\Omega_0, i_0, \omega_0)$  the classical elements of the orbit at the initial time. They can be calculated from the departure perifocal frame taking into account the initial values of  $\vec{x}_0$  and  $\vec{v}_0$ .

Remember that the departure perifocal frame is defined by:

$$\vec{i}_p = \frac{1}{e_0} \vec{e}_0, \quad \vec{j}_p = -\frac{1}{h_0} \vec{h}_0, \quad \vec{k}_p = \vec{i}_p \times \vec{j}_p$$

and from this unit vectors the matrix  $P(0)$  can be calculated. Then, from  $P(0)$  the values of  $(\Omega, i, \omega)$  can be determined. Finally, the values of  $(\varepsilon_1^0(0), \varepsilon_2^0(0), \varepsilon_3^0(0), \varepsilon_4^0(0))$  can be obtained via equations (6-9). It is possible also to perform a **direct calculation** of  $(\varepsilon_1^0(0), \varepsilon_2^0(0), \varepsilon_3^0(0), \varepsilon_4^0(0))$  from the orthogonal matrix  $P(0)$ .

Once the unit vectors  $(\vec{i}_p, \vec{j}_p, \vec{k}_p)$  are known the value of the initial true anomaly  $\sigma_0$  is given by:

$$\cos \sigma_0 = \frac{\vec{x}_0 \cdot \vec{i}_p}{r_0 L_c} = \vec{i} \cdot \vec{i}_p, \quad \sin \sigma_0 = \frac{\vec{j}_p \cdot (\vec{x}_0 \times \vec{i}_p)}{r_0 L_c} = [\vec{j}_p, \vec{i}, \vec{i}_p] = \vec{i} \cdot \vec{k}_p$$

In this analysis we assume than the initial osculating orbit is not circular, that is,  $e_0 \neq 0$ . For the circular orbit the determination of the departure perifocal frame is a bit more involved, as we show later on.

As a consequence of this analysis the initial conditions from which the DROMO equations should be integrated are:

$$\text{at } \sigma = \sigma_0 : \quad \tau = 0, \quad q_1 = q_1(0), \quad q_2 = 0, \quad q_3 = q_3(0), \quad \varepsilon_1^0 = C_1, \quad \varepsilon_2^0 = C_2, \quad \varepsilon_3^0 = C_3, \quad \varepsilon_4^0 = C_4 \quad (10)$$

where  $(C_1, C_2, C_3, C_4)$  are the components of the quaternion that defines the position of the **departure perifocal frame** at the initial time.

Finally, note that the during the motion the following relations hold:

$$\begin{aligned} \varepsilon_1^0 \frac{d\varepsilon_1^0}{d\sigma} + \varepsilon_2^0 \frac{d\varepsilon_2^0}{d\sigma} + \varepsilon_3^0 \frac{d\varepsilon_3^0}{d\sigma} + \varepsilon_4^0 \frac{d\varepsilon_4^0}{d\sigma} &= 0 \\ -\varepsilon_3^0 \frac{d\varepsilon_1^0}{d\sigma} + \varepsilon_4^0 \frac{d\varepsilon_2^0}{d\sigma} + \varepsilon_1^0 \frac{d\varepsilon_3^0}{d\sigma} - \varepsilon_2^0 \frac{d\varepsilon_4^0}{d\sigma} &= 0 \end{aligned}$$

### Initial circular orbit

The procedure described to obtain the initial values of the DROMO variables fails when  $e_0 = 0$ . The  $Ox_p$  axis should point in the direction of the initial eccentricity vector; at first sight, and since  $\vec{e}_0 = \vec{0}$ , we can situate that axis along any straight line of the orbital plane through the origin  $O$ . However a random choice of the  $Ox_p$  axis provides jumps in the ephemeris provided by the numerical integration. To avoid such jumps a careful selection of the axis should be carried out. If we expand the eccentricity vector around  $t_0$  we have:

$$\vec{e}(t) = \vec{e}_0 + \left. \frac{d\vec{e}}{dt} \right|_{t_0} (t - t_0) + \dots$$

where

$$\frac{d\vec{e}}{dt} = \frac{1}{\mu} \left( \vec{f}_p \times \vec{h} + \vec{v} \times (\vec{x} \times \vec{f}_p) \right)$$

To avoid discontinuities in the ephemeris provided by the numerical integration, the unit vector  $\vec{i}_p(0)$  should be selected in the direction pointed out by the derivative of  $\vec{e}$  which is given by:

$$\left. \frac{d\vec{e}}{d\tau} \right|_{\tau=0} = \{2 a_{py} \vec{i} - a_{px} \vec{k}\} \Big|_{\tau=0}$$

It depends on the perturbation on the initial time.



## IV. Asymptotic solution

The parameter  $\epsilon$  involved in the right hand side of the equations

$$\epsilon = J_2 \left( \frac{R_E}{L_c} \right)^2$$

is small (of the order of  $\approx 10^{-3}$  for the Earth). We know that the gravitational potential has, apart from  $J_2$ , many other zonal harmonics. If we only retain  $J_2$  we are neglecting terms of the order of  $J_3, J_4, \dots$ . All these neglected terms are of the order of  $\approx 10^{-6}$  for the Earth, that is, **are of the order of  $J_2^2$** . Thus when we solve the *main problem* we are obtaining an approximate solution for the real motion of the satellite; **even when the solution of the main problem is exact**, that solution is in fact an approximate solution which contains errors of the order of  $J_2$ .

In this paper we obtain an analytic solution of the *main problem* by expanding in terms of the small parameter  $\epsilon$  as follows:

$$\begin{aligned}\tau &= \tau^{(0)} + \epsilon \tau^{(1)} + \dots \\ q_1 &= q_1^{(0)} + \epsilon q_1^{(1)} + \dots \\ q_2 &= q_2^{(0)} + \epsilon q_2^{(1)} + \dots \\ q_3 &= q_3^{(0)} + \epsilon q_3^{(1)} + \dots \\ \varepsilon_1^0 &= \eta_1^{(0)} + \epsilon \eta_1^{(1)} \dots \\ \varepsilon_2^0 &= \eta_2^{(0)} + \epsilon \eta_2^{(1)} \dots \\ \varepsilon_3^0 &= \eta_3^{(0)} + \epsilon \eta_3^{(1)} \dots \\ \varepsilon_4^0 &= \eta_4^{(0)} + \epsilon \eta_4^{(1)} \dots\end{aligned}$$

The goal is to determine the two first terms of the expansion (order unity and order  $\epsilon$ ). This approximate solution will contain errors of the order of  $J_2^2$ . From the point of view of the accuracy, this approximate solution is as good as the **exact** solution of the main problem because both solutions —exact and approximate— involve errors of the order of  $J_2^2$ .

### Order zero solution

The equations for the solution of order zero are quite simple. All the DROMO state variable, except the non-dimensional time  $\tau$ , keep constant values. As a consequence this solution turns out to be:

$$\begin{aligned}q_1^{(0)}(\sigma) &= e_0 q_{3_0}, & q_2^{(0)}(\sigma) &= 0, & q_3^{(0)}(\sigma) &= q_{3_0} \\ \eta_1^{(0)}(\sigma) &= C_1, & \eta_2^{(0)}(\sigma) &= C_2, & \eta_3^{(0)}(\sigma) &= C_3, & \eta_4^{(0)}(\sigma) &= C_4\end{aligned}$$

where, for the sake of brevity, we use the notation:  $q_3(0) = q_{3_0}$ .

Regarding the equation for  $\tau^{(0)}$ , it takes the form:

$$\frac{d\tau^{(0)}}{d\sigma} = \frac{1}{q_{3_0}^3} \frac{1}{(1 + e_0 \cos \sigma)^2}$$

where we made use of the relation:  $q_{3_0} = 1/\psi_0$ . This equation should be integrated with the following initial condition:

$$\tau^{(0)}(0) = 0$$

Through the change of variable:

$$\tan \frac{\sigma}{2} = \sqrt{\frac{1+e_0}{1-e_0}} \tan \frac{u}{2}$$

the equation for  $\tau^{(0)}$  becomes

$$d\tau^{(0)} = \frac{1}{q_{3_0}^3 (1 - e_0^2)^{\frac{3}{2}}} (1 - e_0 \cos u) du$$

and it can be integrated. The solution is:

$$\tau^{(0)}(u) = \frac{1}{q_{3_0}^3 (1 - e_0^2)^{\frac{3}{2}}} [u - u_0 - e_0(\sin u - \sin u_0)]$$

where  $u_0$  is given by

$$\tan \frac{u_0}{2} = \sqrt{\frac{1 - e_0}{1 + e_0}} \tan \frac{\sigma_0}{2}$$

The variable  $u$  coincides with the eccentric anomaly and the solution turns out to be the classical Kepler equation (characteristic of the elliptical motion).

If we start from the pericenter, then  $\sigma_0 = 0$ , the initial eccentric anomaly  $u_0$  is zero and the solution simplifies to:

$$\tau^{(0)}(u) = \frac{1}{q_{3_0}^3 (1 - e_0^2)^{\frac{3}{2}}} (u - e_0 \sin u)$$

### Solution of order $\epsilon$

In order to obtain the order unity solution we have to distinguish between the variable  $\tau^{(1)}$  and the other variables involved in this order:

$$q_1^{(1)}, \quad q_2^{(1)}, \quad q_3^{(1)}, \quad \eta_1^{(1)}, \quad \eta_2^{(1)}, \quad \eta_3^{(1)}, \quad \eta_4^{(1)} \quad (11)$$

We will start by these last variables; the solution for the time will be carried out in the next section. Note, first of all, that the initial conditions for all the variables listed in (11) are **zero**.

With the help of a symbolic manipulator it can be demonstrated that the structure of the first order governing equations is the same for all the variables of order  $\epsilon$  summarized in (11). Let us consider, as example, the equation governing the time evolution of  $q_1^{(1)}$ ; it turns out to be:

$$\frac{dq_1^{(1)}}{d\sigma} = A_0 + \sum_{k=1}^5 [A_k \cos(k\sigma) + B_k \sin(k\sigma)]$$

where the coefficients  $A_0$  y  $A_k, B_k, k = 1, \dots, 5$  are only functions of the initial conditions:  $q_{3_0}, e_0, C_1, C_2, C_3, C_4$ . Since this differential equation must be integrated from zero initial condition:

$$\text{at } \sigma = \sigma_0 : \quad q_1^{(1)}(0) = 0$$

the solution is:

$$q_1^{(1)}(\sigma) = A_0 (\sigma - \sigma_0) + \sum_{k=1}^5 \left[ \frac{A_k}{k} (\sin(k\sigma) - \sin(k\sigma_0)) + \frac{B_k}{k} (\cos(k\sigma) - \cos(k\sigma_0)) \right]$$

An identical situation holds for all the variables listed in (11). Thus we obtain the solution:

$$\begin{pmatrix} q_1^{(1)}(\sigma) \\ q_2^{(1)}(\sigma) \\ q_3^{(1)}(\sigma) \\ \eta_1^{(1)}(\sigma) \\ \eta_2^{(1)}(\sigma) \\ \eta_3^{(1)}(\sigma) \\ \eta_4^{(1)}(\sigma) \end{pmatrix} = \mathcal{A} \begin{pmatrix} 1 \\ \sigma - \sigma_0 \\ \cos(\sigma) - \cos(\sigma_0) \\ \cos(2\sigma) - \cos(2\sigma_0) \\ \cos(3\sigma) - \cos(3\sigma_0) \\ \cos(4\sigma) - \cos(4\sigma_0) \\ \cos(5\sigma) - \cos(5\sigma_0) \\ \sin(\sigma) - \sin(\sigma_0) \\ \sin(2\sigma) - \sin(2\sigma_0) \\ \sin(3\sigma) - \sin(3\sigma_0) \\ \sin(4\sigma) - \sin(4\sigma_0) \\ \sin(5\sigma) - \sin(5\sigma_0) \end{pmatrix}, \quad \mathcal{A} = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \dots & A_{1,11} & A_{1,12} \\ A_{2,1} & A_{2,2} & A_{2,3} & \dots & A_{2,11} & A_{2,12} \\ A_{3,1} & A_{3,2} & A_{3,3} & \dots & A_{3,11} & A_{3,12} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{6,1} & A_{6,2} & A_{6,3} & \dots & A_{6,11} & A_{6,12} \\ A_{7,1} & A_{7,2} & A_{7,3} & \dots & A_{7,11} & A_{7,12} \end{pmatrix} \quad (12)$$

where the elements of the rectangular matrix  $\mathcal{A}$  —of size  $7 \times 12$ — depend, only, of the initial conditions, and can be calculated with the help of a symbolic manipulator (the first column of matrix  $\mathcal{A}$  vanish). The values obtained with Maple has been summarized in Appendix A at the end of the paper.

### Solution for $\tau^{(1)}$

The equation for  $\tau^{(1)}$  takes the form:

$$\frac{d\tau^{(1)}}{d\sigma} = -\frac{H_0 + H_1 \cos \sigma + H_2 \sin \sigma + H_3 \cos(2\sigma) + H_4 \sin(2\sigma) + e_0 H_5 \sigma \sin \sigma}{16(1 + e_0 \cos \sigma)^3}$$

where the coefficients  $H_0, H_1, H_2, H_3, H_4$  and  $H_5$  are functions of  $q_{3_0}$ , the initial eccentricity  $e_0$  and the initial values ( $C_1, C_2, C_3, C_4$ ) of the Euler parameters that define the departure perifocal frame at the initial time. The values of these coefficients are:

$$\begin{aligned} H_0 &= q_{3_0} [-96(1 + e_0)(3 + e_0)(C_1 C_3 - C_2 C_4)^2 + 48(3 - 4e_0 + 2e_0^2)(C_1 C_4 + C_2 C_3)^2 + 12(1 + 4e_0)] \\ H_1 &= q_{3_0} [+64(1 + e_0)(5 + e_0)(C_1 C_3 - C_2 C_4)^2 - 64(1 - e_0)(2 - e_0)(C_1 C_4 + C_2 C_3)^2 - 16(1 + 3e_0)] \\ H_2 &= -64q_{3_0}(1 + e_0)(1 + 2e_0)(C_1 C_3 - C_2 C_4)[1 - 2(C_1^2 + C_2^2)] \\ H_3 &= q_{3_0} [-32(1 - e_0^2)(C_1 C_3 - C_2 C_4)^2 - 16(1 + 2e_0^2)(C_1 C_4 + C_2 C_3)^2 + 4] \\ H_4 &= -16q_{3_0}[1 - 2(C_1^2 + C_2^2)](1 - e_0^2)(C_1 C_3 - C_2 C_4) \\ H_5 &= -24q_{3_0}[1 - 12(C_1 C_4 + C_2 C_3)^2] \end{aligned}$$

The direct integration provides the solution:

$$\tau^{(1)} = -\int_{\sigma_0}^{\sigma} \frac{H_0 + H_1 \cos \sigma + H_2 \sin \sigma + H_3 \cos(2\sigma) + H_4 \sin(2\sigma) + e_0 H_5 \sigma \sin \sigma}{16(1 + e_0 \cos \sigma)^3} d\sigma$$

which can be expressed in a more appropriate form by using the following function:

$$g(\chi) = -\int_0^{\chi} \frac{H_0 + H_1 \cos \sigma + H_2 \sin \sigma + H_3 \cos(2\sigma) + H_4 \sin(2\sigma) + e_0 H_5 \sigma \sin \sigma}{16(1 + e_0 \cos \sigma)^3} d\sigma$$

In terms of  $g(\chi)$  the solution for  $\tau^{(1)}$  is simply:

$$\tau^{(1)} = g(\sigma) - g(\sigma_0)$$

The determination of function  $g(\sigma)$  requires the calculation of 6 integrals, each one of them associated with the 6 coefficients  $H_0, H_1, \dots, H_5$ . All these integrals can be calculated by performing a change of variable. We introduce a pseudo-eccentric anomaly  $u$  defined by:

$$\begin{aligned} \cos \sigma &= \frac{\cos u - e_0}{1 - e_0 \cos u}, & \sin \sigma &= \frac{\sin u \sqrt{1 - e_0^2}}{1 - e_0 \cos u}, & d\sigma &= \frac{\sqrt{1 - e_0^2}}{1 - e_0 \cos u} du \\ \cos u &= \frac{e_0 + \cos \sigma}{1 + e_0 \cos \sigma}, & \sin u &= \frac{\sin \sigma \sqrt{1 - e_0^2}}{1 + e_0 \cos \sigma} \end{aligned}$$

Note that the variable  $u$  is not the eccentric anomaly since  $\sigma$  is not the true anomaly; both variables are the eccentric and true anomalies for the pure Keplerian motion, but no for the perturbed motion.

This way we obtain the following functions:

$$\begin{aligned}
g_0(\sigma) &= -\frac{H_0}{16} \int_0^\sigma \frac{d\sigma}{(1 + e_0 \cos \sigma)^3} = -\frac{H_0}{16} \cdot \frac{1}{(1 - e_0^2)^{5/2}} \cdot \left\{ \frac{1}{2}(2 + e_0^2)u - 2e_0 \sin u + \frac{1}{4}e_0^2 \sin(2u) \right\} \\
g_1(\sigma) &= -\frac{H_1}{16} \int_0^\sigma \frac{\cos \sigma d\sigma}{(1 + e_0 \cos \sigma)^3} = -\frac{H_1}{16} \cdot \frac{1}{(1 - e_0^2)^{5/2}} \cdot \left\{ -\frac{3}{2}e_0 u + (1 + e_0^2) \sin u - \frac{1}{4}e_0 \sin(2u) \right\} \\
g_2(\sigma) &= -\frac{H_2}{16} \int_0^\sigma \frac{\sin \sigma d\sigma}{(1 + e_0 \cos \sigma)^3} = -\frac{H_2}{16} \cdot \frac{1}{(1 - e_0^2)^2} \cdot \left\{ 1 - \cos u + \frac{1}{4}e_0(\cos(2u) - 1) \right\} \\
g_3(\sigma) &= -\frac{H_3}{16} \int_0^\sigma \frac{\cos(2\sigma) d\sigma}{(1 + e_0 \cos \sigma)^3} = -\frac{H_3}{16} \cdot \frac{1}{(1 - e_0^2)^{5/2}} \cdot \left\{ \frac{3}{2}e_0^2 u - 2e_0 \sin u + \frac{1}{4}(2 - e_0^2) \sin(2u) \right\} \\
g_4(\sigma) &= -\frac{H_4}{16} \int_0^\sigma \frac{\sin(2\sigma) d\sigma}{(1 + e_0 \cos \sigma)^3} = -\frac{H_4}{16} \cdot \frac{1}{(1 - e_0^2)^2} \cdot \left\{ 2e_0(\cos u - 1) + \frac{1}{2}(1 - \cos(2u)) \right\} \\
g_5(\sigma) &= -\frac{H_5}{16} \int_0^\sigma \frac{e_0 \sigma \sin \sigma d\sigma}{(1 + e_0 \cos \sigma)^3} = -\frac{H_5}{16} \cdot \frac{1}{2} \cdot \left\{ \frac{\sigma}{(1 + e_0 \cos \sigma)^2} - \int_0^\sigma \frac{d\sigma}{(1 + e_0 \cos \sigma)^2} \right\} = \\
&= -\frac{H_5}{16} \cdot \frac{1}{2} \cdot \frac{1}{(1 - e_0^2)^{3/2}} \left\{ \sigma(u) \frac{(1 - e_0 \cos u)^2}{(1 - e_0^2)^{1/2}} - u + e_0 \sin u \right\}
\end{aligned}$$

from which it is possible to determine the function  $g(\sigma)$ :

$$g(\sigma) = \sum_{i=0}^5 g_i(\sigma)$$

## V. Mean values

The effects of the  $J_2$  harmonic on an elliptical orbit are very well known. The main contributions turn out to be the **regression of nodes** and the **advance of the perigee**. The classical expression for the averaged values of these two effects are:

$$\left\langle \frac{d\Omega}{dt} \right\rangle = -\frac{3}{2} J_2 \frac{R_E^2}{a^2} \frac{n \cos i}{(1 - e^2)^2} \quad (13)$$

$$\left\langle \frac{d\omega}{dt} \right\rangle = \frac{3}{4} J_2 \frac{R_E^2}{a^2} \frac{n(4 - 5 \sin^2 i)}{(1 - e^2)^2} \quad (14)$$

Both expressions can be deduced easily from the asymptotic solution obtained in the previous sections. For the sake of brevity we do not present here the analysis that provides these averaged derivatives from the asymptotic solution. However, we remember these values for describe more precisely the checking process carried out in the following section.

## VI. Checking the solution

In order to check the asymptotic solution obtained in the previous sections we selected a retrograde orbit defined by the following classical elements:

$$\Omega = 1.4789153937 \text{ rad}, \quad \omega = 0.6154797087 \text{ rad}, \quad i = 120^\circ, \quad e_0 = 0.5625, \quad q_{30} = 0.8 \quad (15)$$

or equivalent, defined by the following state vector:

$$\begin{aligned}
x_1(0) &= 2568.0678221016 \text{ km}, & \dot{x}_1(0) &= +3.3144253378 \text{ km/s} \\
y_1(0) &= 5574.2514415857 \text{ km}, & \dot{y}_1(0) &= -5.7407530829 \text{ km/s} \\
z_1(0) &= 3543.4094444444 \text{ km}, & \dot{z}_1(0) &= +6.6288506755 \text{ km/s}
\end{aligned}$$

For this orbit the small parameter  $\epsilon$  takes the value:

$$\epsilon = 0.000876927627$$



To check the asymptotic solution, an integration of the equations between  $\sigma = 0$  and  $\sigma = 8\pi$  has been performed. The selection of this orbit is completely arbitrary; in fact, this orbit appeared in our group in a different context, and we decided to use it in this paper in order to check the goodness of the asymptotic solution.

Figure 3 summarizes the comparison between the results obtained with the asymptotic solution and the numerical solution of the equations for the initial conditions given in (15). Both solutions are practically the same for the, approximately, 4 orbits of the simulation.

Figure 4 summarizes the comparison between the results obtained with the asymptotic solution and the solution obtained by integrating the equations of motion using the Cowell method (for the same initial conditions given in (15)). Figures shows the time evolution of the inertial coordinates and velocities calculated for the above perturbed orbit. It is not possible to distinguish both solution during the first 20 orbits.

It should be noticed that the error of the asymptotic solution is of the order  $J_2^2$ ; since the asymptotic solution shows secular terms, the error associated with such terms increases when the ideal anomaly  $\sigma$  increases; **thus the asymptotic solution fails when  $\sigma \approx 1/J_2$** . Due to the small value of  $J_2$  this approximated solution provide good results during 7 or 10 days, approximately.

The asymptotic solution can be improved with a rectification in each orbit; however, we do not describe here such a process that permit to extend the validity of the solution; in future works we will face this problem.

Figure 5 shows the time evolution of two classical elements: the right ascension of the ascending node,  $\Omega$  and the argument of perigee  $\omega$ . Notice that the asymptotic solution provides the **osculating values** of these classical elements; in the figure the osculating values are compared with the mean values obtained from the classical expressions summarized in equations (13-14).

## VII. Conclusions

In this paper an asymptotic solution of the main problem, in terms of DROMO variables, has been obtained. This approximate solution is, in fact, as good as an exact solution of the governing equations of the main problem, if we consider that in the gravitational potential of the Earth there are more harmonics, apart from  $J_2$ , which are of the same order than  $J_2^2$  or smaller.

The solution provides the osculating values of the classical elements and from the point of view of the calculations is very fast and reliable. The validity of such a solution is or the order of 7 or 10 days, because due to secular terms the asymptotic expansions fails when  $\sigma \approx 1/J_2 \approx 1000$ .

In future works we will show how it is possible to increase the range of validity of the solution by re-normalizing the solution in each orbit.

## VIII. Acknowledgements

This work is part of the research project entitled **Dynamic simulation of complex space systems** (AYA2010-18796) supported by the DGI of the Spanish Ministry of Science and Innovation.

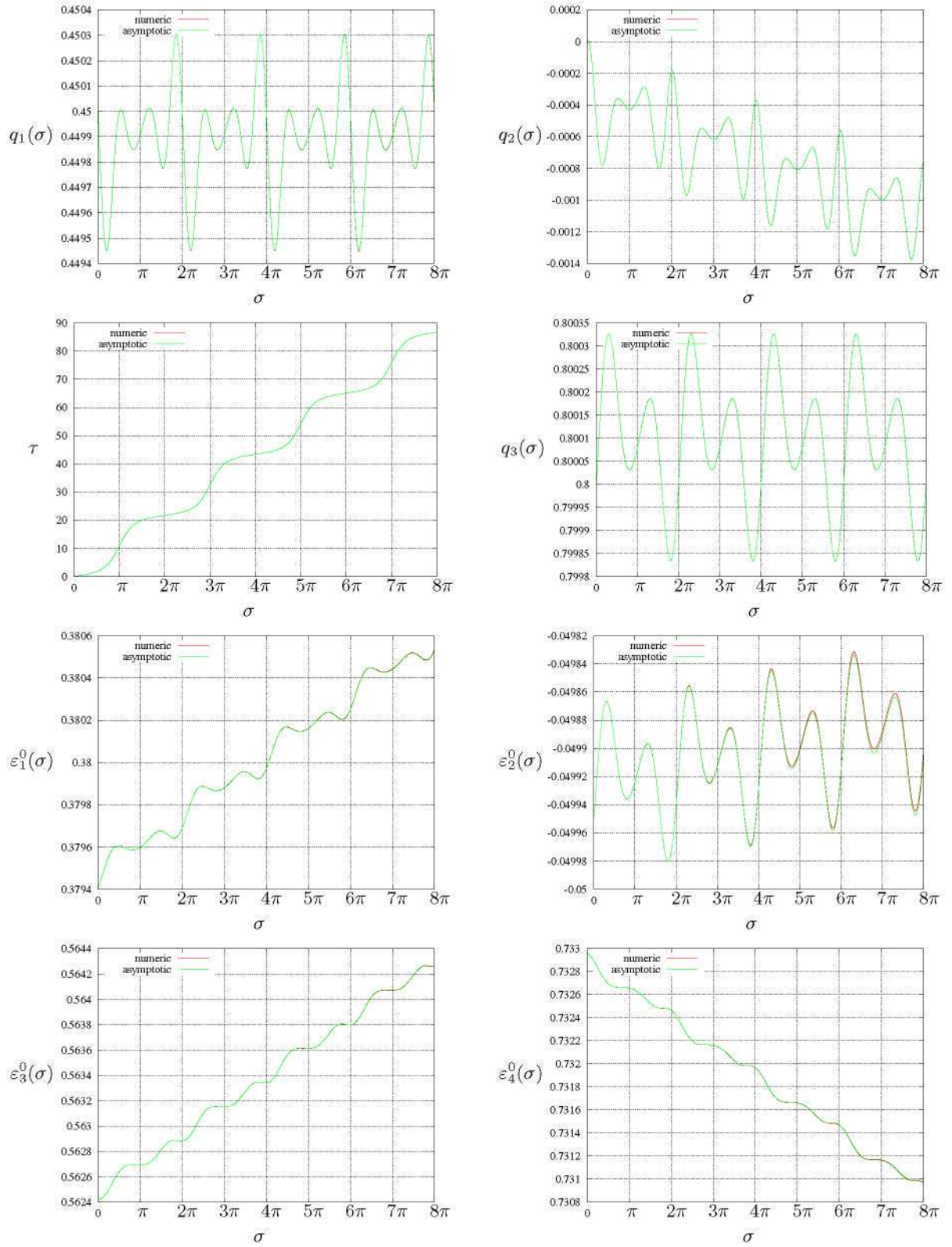


Figure 3: Comparison between the results obtained with the asymptotic solution and the **numerical integration** of the DROMO equations for the initial conditions given in (15)

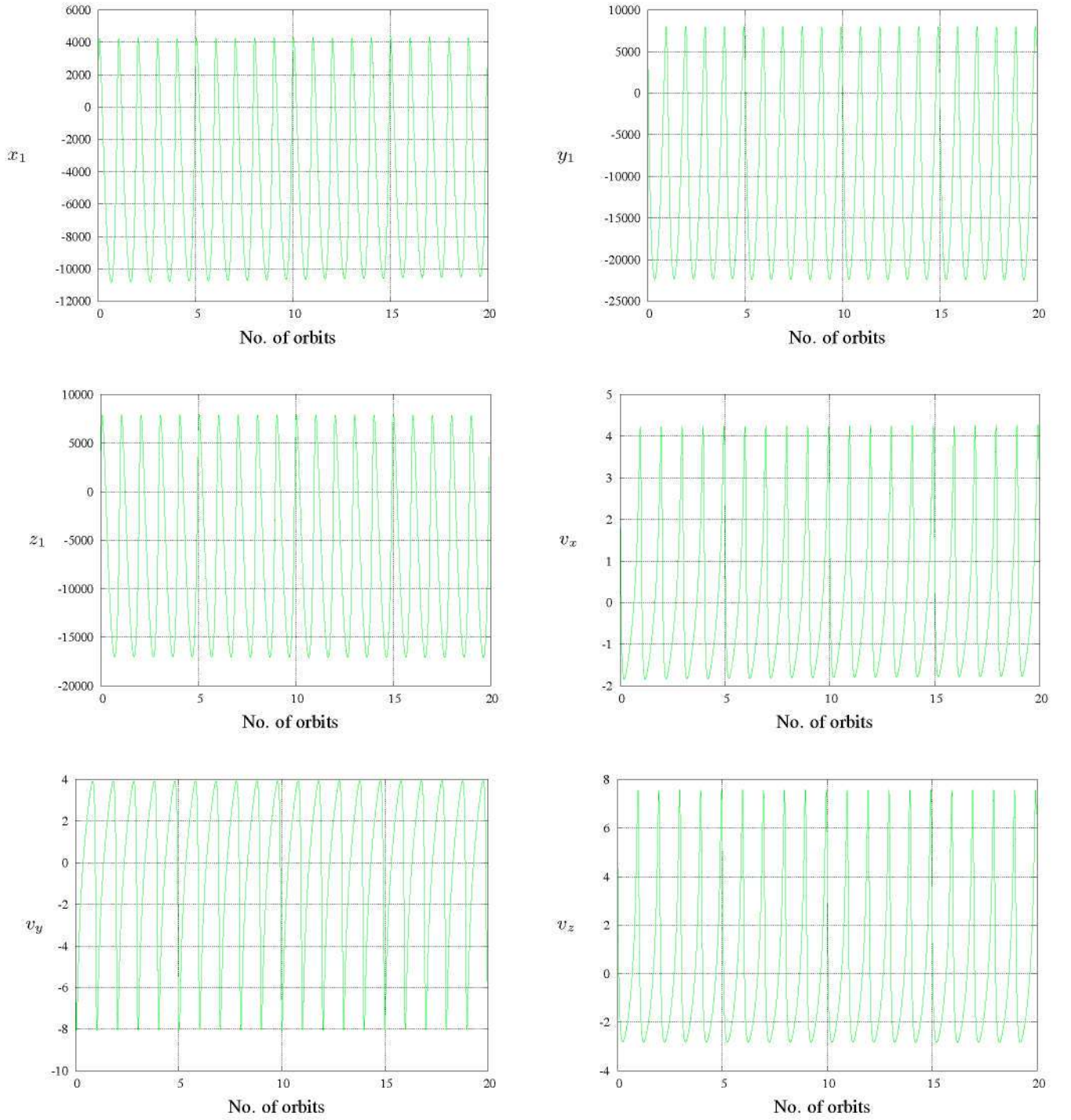


Figure 4: Comparison between the results obtained with the asymptotic solution and the direct numerical integration of the equations using the Cowell method for the initial conditions given in (15). This figure shows the time evolution of the inertial positions and velocities. Distances in km, velocities in km/s

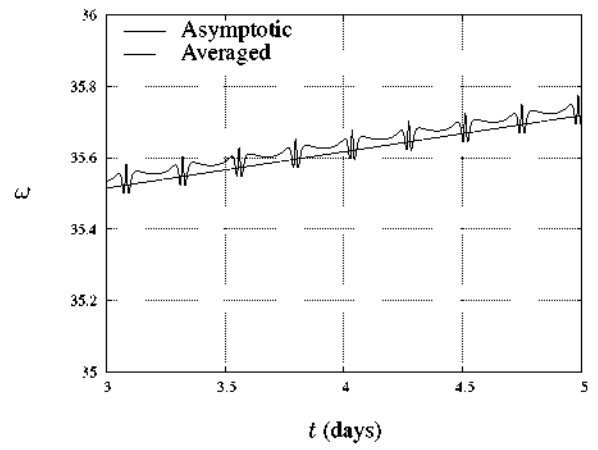
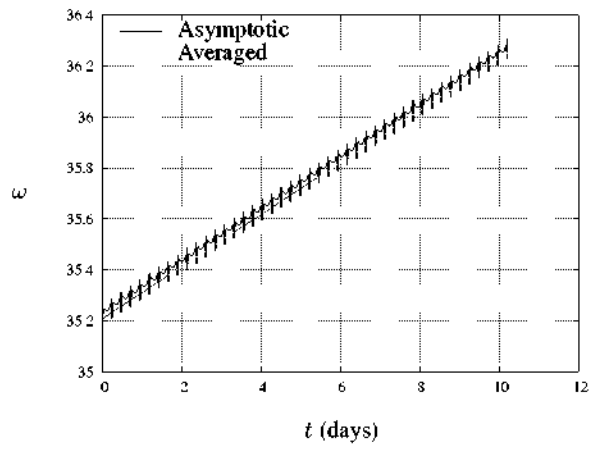
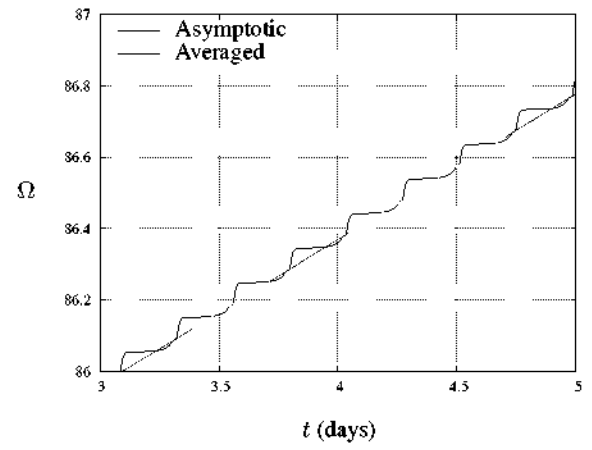
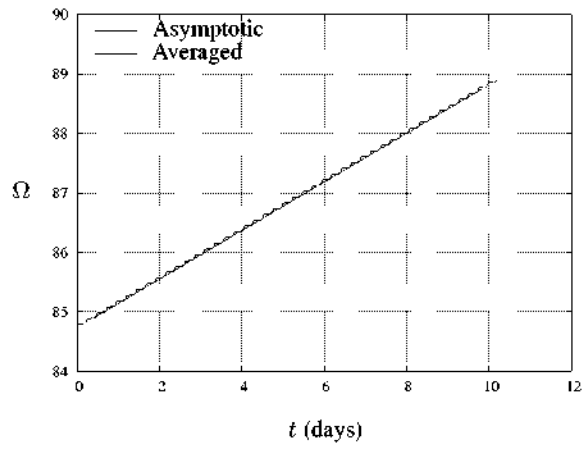


Figure 5: Classical elements  $\Omega$  and  $\omega$  as functions of time



## IX. Appendix A: elements of matrix $\mathcal{A}$

The elements of the matrix  $\mathcal{A}$  are functions of the initial values  $q_{3_0}, e_0, C_1, C_2, C_3, C_4$  and they take the values:

$$\begin{aligned}
 \mathcal{A}_{1,2} &= 0 \\
 \mathcal{A}_{1,3} &= \frac{3}{16} q_{3_0}^5 \{ -16 (e_0^2 + 1) (C_1 C_3 - C_2 C_4)^2 + 4 (e_0^2 + 10) (C_1 C_4 + C_2 C_3)^2 + (e_0^2 - 2) \} \\
 \mathcal{A}_{1,4} &= \frac{3}{4} q_{3_0}^5 e_0 \{ 1 - 12 (C_1 C_3 - C_2 C_4)^2 \} \\
 \mathcal{A}_{1,5} &= \frac{1}{32} q_{3_0}^5 \{ -8 (9 e_0^2 + 28) (C_1 C_3 - C_2 C_4)^2 - 4 (28 + 3 e_0^2) (C_1 C_4 + C_2 C_3)^2 + 7 (e_0^2 + 4) \} \\
 \mathcal{A}_{1,6} &= \frac{9}{16} q_{3_0}^5 e_0 \{ 1 - 8 (C_1 C_3 - C_2 C_4)^2 - 4 (C_1 C_4 + C_2 C_3)^2 \} \\
 \mathcal{A}_{1,7} &= \frac{e_0}{6} \mathcal{A}_{1,6} = \frac{3}{32} q_{3_0}^5 e_0^2 \{ 1 - 8 (C_1 C_3 - C_2 C_4)^2 - 4 (C_1 C_4 + C_2 C_3)^2 \} \\
 \mathcal{A}_{1,8} &= -\frac{3}{4} q_{3_0}^5 (2 + e_0^2) (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{1,9} &= -\frac{9}{2} q_{3_0}^5 e_0 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{1,10} &= -\frac{1}{8} q_{3_0}^5 (28 + 9 e_0^2) (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{1,11} &= \frac{1}{2} \mathcal{A}_{1,9} = -\frac{9}{4} q_{3_0}^5 e_0 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{1,12} &= \frac{e_0}{12} \mathcal{A}_{1,9} = -\frac{3}{8} q_{3_0}^5 e_0^2 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2)
 \end{aligned}$$

---


$$\begin{aligned}
 \mathcal{A}_{2,2} &= \frac{3}{4} q_{3_0}^5 e_0 \{ 12 (C_1 C_4 + C_2 C_3)^2 - 1 \} \\
 \mathcal{A}_{2,3} &= \frac{3}{4} q_{3_0}^5 (3 e_0^2 - 2) (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{2,4} &= \frac{9}{2} q_{3_0}^5 e_0 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{2,5} &= \frac{1}{8} q_{3_0}^5 (11 e_0^2 + 28) (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{2,6} &= \frac{1}{2} \mathcal{A}_{2,4} = \frac{9}{4} q_{3_0}^5 e_0 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{2,7} &= \frac{e_0}{6} \mathcal{A}_{2,6} = \frac{3}{8} q_{3_0}^5 e_0^2 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
 \mathcal{A}_{2,8} &= \frac{3}{16} q_{3_0}^5 \{ -16 (2 e_0^2 - 1) (C_1 C_3 - C_2 C_4)^2 + 4 (5 e_0^2 + 14) (C_1 C_4 + C_2 C_3)^2 + (e_0^2 - 6) \} \\
 \mathcal{A}_{2,9} &= \frac{3}{4} q_{3_0}^5 e_0 \{ 1 - 12 (C_1 C_3 - C_2 C_4)^2 \} \\
 \mathcal{A}_{2,10} &= \frac{1}{32} q_{3_0}^5 \{ -8 (11 e_0^2 + 28) (C_1 C_3 - C_2 C_4)^2 - 4 (5 e_0^2 + 28) (C_1 C_4 + C_2 C_3)^2 + (9 e_0^2 + 28) \} \\
 \mathcal{A}_{2,11} &= \frac{9}{16} q_{3_0}^5 e_0 \{ 1 - 8 (C_1 C_3 - C_2 C_4)^2 - 4 (C_1 C_4 + C_2 C_3)^2 \} \\
 \mathcal{A}_{2,12} &= \frac{e_0}{6} \mathcal{A}_{2,11} = \frac{3}{32} q_{3_0}^5 e_0^2 \{ 1 - 8 (C_1 C_3 - C_2 C_4)^2 - 4 (C_1 C_4 + C_2 C_3)^2 \}
 \end{aligned}$$


---

$$\begin{aligned}
\mathcal{A}_{3,2} &= 0 \\
\mathcal{A}_{3,3} &= \frac{3}{4} q_{3_0}^5 e_0 \{8 (C_1 C_3 - C_2 C_4)^2 + 4 (C_1 C_4 + C_2 C_3)^2 - 1\} \\
\mathcal{A}_{3,4} &= \frac{3}{4} q_{3_0}^5 \{8 (C_1 C_3 - C_2 C_4)^2 + 4 (C_1 C_4 + C_2 C_3)^2 - 1\} \\
\mathcal{A}_{3,5} &= \frac{1}{4} q_{3_0}^5 e_0 \{8 (C_1 C_3 - C_2 C_4)^2 + 4 (C_1 C_4 + C_2 C_3)^2 - 1\} \\
\mathcal{A}_{3,6} &= 0 \\
\mathcal{A}_{3,7} &= 0 \\
\mathcal{A}_{3,8} &= 3 q_{3_0}^5 e_0 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
\mathcal{A}_{3,9} &= 3 q_{3_0}^5 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
\mathcal{A}_{3,10} &= q_{3_0}^5 e_0 (C_1 C_3 - C_2 C_4) (1 - 2 C_2^2 - 2 C_1^2) \\
\mathcal{A}_{3,11} &= 0 \\
\mathcal{A}_{3,12} &= 0
\end{aligned}$$


---

$$\begin{aligned}
\mathcal{A}_{4,2} &= \frac{3}{2} q_{3_0}^4 (C_1 C_4 + C_2 C_3) (2 C_3 C_4 C_1 + 2 C_3^2 C_2 - C_2) \\
\mathcal{A}_{4,3} &= \frac{3}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_4 C_1^2 - C_4 + 4 C_4 C_2^2 - 2 C_2 C_3 C_1) \\
\mathcal{A}_{4,4} &= \frac{3}{4} q_{3_0}^4 (C_1 C_4 + C_2 C_3) (2 C_4 C_1^2 - C_4 + 4 C_4 C_2^2 - 2 C_2 C_3 C_1) \\
\mathcal{A}_{4,5} &= \frac{1}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_4 C_1^2 - C_4 + 4 C_4 C_2^2 - 2 C_2 C_3 C_1) \\
\mathcal{A}_{4,6} &= 0 \\
\mathcal{A}_{4,7} &= 0 \\
\mathcal{A}_{4,8} &= \frac{3}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (6 C_3 C_4 C_1 - C_2 + 2 C_3^2 C_2 - 4 C_2 C_4^2) \\
\mathcal{A}_{4,9} &= \frac{3}{4} q_{3_0}^4 (C_1 C_4 + C_2 C_3) (2 C_3 C_4 C_1 + C_2 - 2 C_3^2 C_2 - 4 C_2 C_4^2) \\
\mathcal{A}_{4,10} &= \frac{1}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_3 C_4 C_1 + C_2 - 2 C_3^2 C_2 - 4 C_2 C_4^2) \\
\mathcal{A}_{4,11} &= 0 \\
\mathcal{A}_{4,12} &= 0
\end{aligned}$$


---

$$\begin{aligned}
\mathcal{A}_{5,2} &= -\frac{3}{2} q_{3_0}^4 (C_1 C_4 + C_2 C_3) (2 C_2 C_4 C_3 + 2 C_1 C_4^2 - C_1) \\
\mathcal{A}_{5,3} &= \frac{3}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (-2 C_1 C_4 C_2 + 4 C_1^2 C_3 - C_3 + 2 C_2^2 C_3) \\
\mathcal{A}_{5,4} &= \frac{3}{4} q_{3_0}^4 (C_1 C_4 + C_2 C_3) (-2 C_1 C_4 C_2 + 4 C_1^2 C_3 - C_3 + 2 C_2^2 C_3) \\
\mathcal{A}_{5,5} &= \frac{1}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (-2 C_1 C_4 C_2 + 4 C_1^2 C_3 - C_3 + 2 C_2^2 C_3) \\
\mathcal{A}_{5,6} &= 0 \\
\mathcal{A}_{5,7} &= 0 \\
\mathcal{A}_{5,8} &= \frac{3}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (-2 C_1 C_4^2 - 6 C_2 C_4 C_3 + 4 C_1 C_3^2 + C_1) \\
\mathcal{A}_{5,9} &= \frac{3}{4} q_{3_0}^4 (C_1 C_4 + C_2 C_3) (2 C_1 C_4^2 - 2 C_2 C_4 C_3 + 4 C_1 C_3^2 - C_1) \\
\mathcal{A}_{5,10} &= \frac{1}{4} q_{3_0}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_1 C_4^2 - 2 C_2 C_4 C_3 + 4 C_1 C_3^2 - C_1) \\
\mathcal{A}_{5,11} &= 0 \\
\mathcal{A}_{5,12} &= 0
\end{aligned}$$

---


$$\begin{aligned}
\mathcal{A}_{6,2} &= -\frac{3}{2} q_{30}^4 (C_1 C_4 + C_2 C_3) (2 C_2 C_3 C_1 - C_4 + 2 C_4 C_1^2) \\
\mathcal{A}_{6,3} &= \frac{3}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (-2 C_3 C_4 C_1 + 4 C_2 C_4^2 - C_2 + 2 C_3^2 C_2) \\
\mathcal{A}_{6,4} &= \frac{3}{4} q_{30}^4 (C_1 C_4 + C_2 C_3) (-2 C_3 C_4 C_1 + 4 C_2 C_4^2 - C_2 + 2 C_3^2 C_2) \\
\mathcal{A}_{6,5} &= \frac{1}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (-2 C_3 C_4 C_1 + 4 C_2 C_4^2 - C_2 + 2 C_3^2 C_2) \\
\mathcal{A}_{6,6} &= 0 \\
\mathcal{A}_{6,7} &= 0 \\
\mathcal{A}_{6,8} &= \frac{3}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (-2 C_4 C_1^2 + C_4 + 4 C_4 C_2^2 - 6 C_2 C_3 C_1) \\
\mathcal{A}_{6,9} &= \frac{3}{4} q_{30}^4 (C_1 C_4 + C_2 C_3) (2 C_4 C_1^2 - C_4 + 4 C_4 C_2^2 - 2 C_2 C_3 C_1) \\
\mathcal{A}_{6,10} &= \frac{1}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_4 C_1^2 - C_4 + 4 C_4 C_2^2 - 2 C_2 C_3 C_1) \\
\mathcal{A}_{6,11} &= 0 \\
\mathcal{A}_{6,12} &= 0
\end{aligned}$$


---

$$\begin{aligned}
\mathcal{A}_{7,2} &= \frac{3}{2} q_{30}^4 (C_1 C_4 + C_2 C_3) (2 C_1 C_4 C_2 + 2 C_2^2 C_3 - C_3) \\
\mathcal{A}_{7,3} &= \frac{3}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_1 C_4^2 - 2 C_2 C_4 C_3 + 4 C_1 C_3^2 - C_1) \\
\mathcal{A}_{7,4} &= \frac{3}{4} q_{30}^4 (C_1 C_4 + C_2 C_3) (2 C_1 C_4^2 - 2 C_2 C_4 C_3 + 4 C_1 C_3^2 - C_1) \\
\mathcal{A}_{7,5} &= \frac{1}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_1 C_4^2 - 2 C_2 C_4 C_3 + 4 C_1 C_3^2 - C_1) \\
\mathcal{A}_{7,6} &= 0 \\
\mathcal{A}_{7,7} &= 0 \\
\mathcal{A}_{7,8} &= \frac{3}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (6 C_1 C_4 C_2 - 4 C_1^2 C_3 + 2 C_2^2 C_3 - C_3) \\
\mathcal{A}_{7,9} &= \frac{3}{4} q_{30}^4 (C_1 C_4 + C_2 C_3) (2 C_1 C_4 C_2 - 4 C_1^2 C_3 - 2 C_2^2 C_3 + C_3) \\
\mathcal{A}_{7,10} &= \frac{1}{4} q_{30}^4 e_0 (C_1 C_4 + C_2 C_3) (2 C_1 C_4 C_2 - 4 C_1^2 C_3 - 2 C_2^2 C_3 + C_3) \\
\mathcal{A}_{7,11} &= 0 \\
\mathcal{A}_{7,12} &= 0
\end{aligned}$$

## References

- <sup>1</sup>Deprit, A., "Ideal elements for perturbed Keplerian motions," *Journal of Research of the National Bureau of Standards. Sec. B: Math. Sci.*, Vol. 79B, No. 1-2, 1975, pp. 1–15, Paper 79B1&2–416.
- <sup>2</sup>Deprit, A., "Ideal frames for perturbed Keplerian motions," *Celestial Mechanics and Dynamical Astronomy*, Vol. 13, No. 2, 1976, pp. 253–263.
- <sup>3</sup>Palacios, M., Abad, A., and Elipe, A., "An efficient numerical method for orbit computations," *Astrodynamics 1991*, Vol. 1, 1992, pp. 265–274.
- <sup>4</sup>Palacios, M. and Calvo, C., "Quaternions formulation versus regularization in numerical orbit computation," *Astrodynamics 1993*, 1993, pp. 2629–2644.
- <sup>5</sup>Palacios, M. and Calvo, C., "Ideal frames and regularization in numerical orbit computation," *Journal of the Astronautical Sciences*, Vol. 44, No. 1, 1996, pp. 63–77.
- <sup>6</sup>Cid, R. and San Saturnio, M., "Motion of rigid bodies in a set of redundant variables," *Celestial Mechanics and Dynamical Astronomy*, Vol. 42, No. 1, 1987, pp. 263–277.
- <sup>7</sup>Peláez, J., Hedo, J. M., and Rodríguez, P., "A special perturbation method in orbital Dynamics," *AAS/AIAA Spaceflight Mechanics Meeting 2005*, edited by D. A. Vallado, M. J. Gabor, and P. N. Desai, Vol. 120 of *Advances in the Astronautical Sciences*, 2005, pp. 1061–1078.
- <sup>8</sup>Peláez, J., Hedo, J. M., and de Andrés, P. R., "A special perturbation method in orbital dynamics," *Celestial Mechanics and Dynamical Astronomy*, Vol. 97, 2007, pp. 131–150, 10.1007/s10569-006-9056-3.
- <sup>9</sup>C. Bombardelli, G. B. and Peláez, J., "Asymptotic Solution for the Two-Body Problem with Constant Tangential Thrust Acceleration," *Celestial Mechanics and Dynamical Astronomy*, Vol. 110, No. 3, 2011, pp. 239–256.